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Liquidity Management and Corporate Demand for Hedging and Insurance*

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Liquidity Management and Corporate Demand for Hedging and Insurance

Abstract: We analyze the demand for hedging and insurance by a firm facing cash-flow risks. We study how the firm's liquidity management policy interacts with two types of risk: a Brownian risk that can be hedged through a financial derivative, and a Poisson risk that can be insured by an insurance contract. We find that the patterns of insurance and hedging decisions are pole apart: cash-poor firms should hedge but not insure, whereas the opposite is true for cash-rich firms. We also find non monotonic effects of profitability. This may explain the mixed findings of empirical studies on corporate demand for hedging and insurance.

Key Words: Liquidity Management, Risk Management, Corporate Hedging.

1 Introduction

Corporate risk management has been the subject of a large academic literature in the last thirty years. This literature aims at filling the gap between the irrelevance results derived from the benchmark of perfect capital markets (Modigliani and Miller, 1958) and the practical importance of risk management decisions in modern corporations.

Several directions have been explored for explaining how and why firms should hedge their risks,¹ among which: managerial risk aversion (Stulz, 1984), tax optimization (Smith and Stulz, 1985), cost of financial distress (Smith and Stulz, 1985), cost of external financing (Stulz, 1990; Froot, Scharfstein and Stein, 1993). A few papers have applied these ideas to model corporate demand for insurance.²

The specific testable implications derived from each of these models are different, but some of them are robust. In particular, there is now a consensus among financial economists that profitability and liquidity should be important determinants of firms' hedging and insurance policies. All of the above theories predict indeed that less liquid and less profitable firms should manage their risks more actively. However, this is only partially confirmed by the data. Indeed, the empirical literature (see for example Tufano³ 1996 and Geczy et al.⁴ 1997) finds that liquidity is indeed an important determinant of hedging (more liquid firms hedge less), but there is no clear evidence on the impact of liquidity on insurance decisions. Also, profitability does not seem to have a clear and robust impact on hedging and insurance decisions.

The main objective of this paper is to show that when liquidity management and risk management decisions are endogenized simultaneously, the theoretical impact of prof-

¹For a critical assessment of these ideas, see Smith and Stulz (1985).

²See for example Mayers and Smith (1982), (1987) and (1990), or Caillaud et al. (2000).

³Tufano (1996) studies risk management behavior in the US gold mining industry. He finds that managerial compensation (in the form of share ownership or stock options holdings) is a major determinant of the use of derivatives: when managers own shares, firms hedge more, but when managers own options, firms hedge less. He also finds that more liquid firms hedge less.

⁴Geczy et al. (1997) study a sample of 372 US firms, composed of the Fortune 500 largest firms that have at least one source of foreign exchange exposure. They use a logit model to explore the determinants of the use of currency derivatives. They find evidence that higher quick ratios, indicating more internally available funds, are associated with a lower probability of using currency derivative instruments.

itability is non monotonic: the firms that gain the most from actively managing their risks are not the less profitable nor the most profitable. Moreover, when insurance decisions are explicitly modeled, we find that the optimal patterns of hedging and insurance decisions by firms are exactly opposite: cash-poor firms should hedge but not insure, whereas the opposite is true for cash-rich firms. Thus the relation between liquidity and optimal risk management decisions of firms may be more complex than initially thought. This may explain the mixed findings of empirical studies on corporate demand for hedging and insurance, who typically use linear specifications.

The paper uses a continuous time stationary framework à la Merton (1974) or Leland (1998), with the important difference that we focus on liquidity risk rather than solvency risk.⁵ Namely, we consider a model similar to Radner and Shepp (1996) and Jeanblanc and Shiryaev (1995) where a firm operates a profitable technology but is confronted with unpredictable⁶ liquidity shocks. Cash management is used to reduce the cost of two financial frictions that work in opposite directions: on the one hand issuing new securities is costly and on the other hand, free cash-flows can be wasted by managers. We show that the optimal liquidity management policy of the firm is to accumulate cash balances up to some target level x^* and distribute all further gains as dividends. As we explain below, x^* can be viewed as a measure of the cost of financial frictions.

We first construct a simple model that allows to integrate liquidity management and risk management decisions (Section 2). We then use this model to characterize these optimal decisions (Section 3). Section 4 provides some robustness checks. Section 5 estimates the gains from hedging and insuring. Section 6 concludes by deriving testable implications on the impact of profitability and risk on corporate hedging. Mathematical proofs are in the appendix.

⁵Similar frameworks have been used to analyze the impact of solvency regulations and regulatory audits on banks' portfolio decisions: see e.g. Merton (1978), Bhattacharya et al. (2002) or Dangl and Lehar (2001).

⁶Specifically, in our model instantaneous cash flows contain a Brownian component. Equivalently, cumulated cash flows X_t follow a mixed Poisson-diffusion process. By contrast, in Merton (1978) and Leland (1998) the profitability of the firm is uncertain but cash flows are predictable. That is, in their model the cash flow process X_t satisfies $dX_t = \mu_t dt$ (no diffusion term) but μ_t itself follows a diffusion.

2 Integrating Liquidity Management and Risk Management

One of our objectives is to contrast the behavior of a firm with respect to two types of risk:

- risks like currency risk that continuously impact the earnings of the firm and can be hedged through market instruments like futures contracts,
- risks like industrial catastrophes that have a small probability of occurrence but a large cost if they occur, and can only be covered through an insurance contract⁷.

In order to introduce a need for active cash management, we follow the strategy of Décamps et al. (2008) and introduce two financial frictions: a cost of issuing new securities (which gives a precautionary motive for cash holdings) and an agency cost à la Jensen (1986), implying that cash holdings within the firm have a lower rate of return than the risk free rate because cash can be diverted by managers at their own advantage to the detriment of outside shareholders. This gives the second part of the trade-off, i.e. a reason for limiting these cash holdings. We want to build a model allowing to study the interactions between liquidity management (i.e. when to issue new securities and when to distribute dividends) and risk management (when to hedge and when to insure). For simplicity we rule out other frictions such as taxes (implying that the firm will never issue debt but only equity) or transaction costs on insurance contracts or hedging instruments.

2.1 The Model

We consider a firm characterized by a cash-flow generating process that we decompose into two parts:

⁷In the real world, the situation is more blurred: there are market instruments that cover Poisson risks and vice versa some insurance contracts can cover Brownian risks. Moreover, most risks have both a continuous and a jump components. Our distinction is only made for expository purposes.

- “primary” earnings that result from the core activities of the firm and cannot be modified⁸,
- other profits and losses that can be modified by the firm through several dimensions of its financial policy: dividend payments, issuance of new securities, hedging and insurance decisions.

The firm’s behavior is entirely determined (for a given set of parameters) by a unique state variable, the level X_t of cash holdings. This level of cash holdings is controlled by both the dividend policy Z and the issuance policy I . Therefore, in the absence of hedging or insurance, the dynamics of X_t is given by:

$$dX_t = [\mu dt + \sigma dW_t] + r_0 X_t dt + [\sigma_R dW_t^R - L dP_t] - dZ_t + dI_t - dF_t. \quad (1)$$

The first bracket corresponds to the “primary” earnings of the firm: μ is the expected profitability per unit of time, σ the volatility of “primary” earnings and W_t a standard Wiener process which cannot be hedged nor insured. $r_0 \geq 0$ is the interest rate received on cash holdings. The second bracket corresponds to the two risks that can be managed: the Brownian risk $\sigma_R dW_t^R$ can be hedged at no cost (W_t^R is also a Wiener process, independent from W_t), while the Poisson risk can be insured (P_t is a Poisson process with intensity λ) for a fair premium λL per unit of time. We assume that $\mu > \lambda L$ (otherwise the technology would not be profitable). The last three terms correspond to the financial policy of the firm: Z_t is the (non decreasing) cumulative dividend process, I_t is the (non decreasing) cumulative issuance process and F_t is the cumulative cost of external financing: A fixed cost f is incurred every time new securities⁹ are issued.¹⁰ Therefore, the issuance policy consists of choosing the dates and the amounts of new securities issued. We represent the times of issuance by a sequence $(\tau_n)_{n \in \mathbb{N}}$ and the amounts issued by another sequence

⁸For simplicity we assume that the size of the firm is fixed, and do not introduce investment nor depreciation.

⁹Since there are no taxes nor moral hazard, it can be shown that the optimal security is always equity.

¹⁰We could also introduce proportional issuance costs as in Decamps et al. (2008). This would complicate the analysis without changing the qualitative results.

$(i_n)_{n \in \mathbb{N}}$. I_t is modeled by

$$I_t = \sum_{n=1}^{\infty} i_n \mathbb{1}_{\{\tau_n \leq t\}}, \quad (2)$$

while F_t is modeled by

$$F_t = -f \sum_{n=1}^{\infty} \mathbb{1}_{\{\tau_n \leq t\}}. \quad (3)$$

When its cash position is negative, the firm has the choice between issuing new equity or defaulting, in which case shareholders lose everything. Default time is defined as follows:

$$\tau_B = \inf\{t \geq 0 \mid X_t < 0\}. \quad (4)$$

Note that τ_B can be infinite if the firm decides to issue new equity every time its cash reserves fall to zero.

All investors are risk neutral and discount the future at rate r . Thus the only financial frictions come from the twin assumptions that $f > 0$ (which explains why the firm needs to hold cash) and $r_0 < r$ (which explains why the firm does not retain all earnings).

Given an issuance policy $((\tau_n)_{n \geq 1}, (i_n)_{n \geq 1})$, a dividend policy Z , and current cash reserves $x \geq 0$, the value of the firm along the policy can be computed as:

$$v(x; (\tau_n)_{n \geq 1}, (i_n)_{n \geq 1}, Z) = \mathbb{E}_x \left[\int_0^{\tau_B} e^{-rt} (dZ_t - dI_t) \right], \quad (5)$$

The objective is to find the optimal shareholder value function V , defined as

$$V(x) = \sup_{(\tau_n)_{n \geq 1}, (i_n)_{n \geq 1}, Z} \{v(x; (\tau_n)_{n \geq 1}, (i_n)_{n \geq 1}, Z)\}; \quad x \geq 0. \quad (6)$$

2.2 The Case of Perfect Financial Markets ($f = 0$)

In the absence of issuance costs (i.e. when $f = 0$) the firm would never hold any cash. As long as there is no accident, the firm would either fully distribute its earnings $[\mu dt + \sigma dW_t + \sigma_R dW_t^R]$ in the form of dividends when these earnings are positive or issue new equity when they are negative. It would be totally indifferent concerning its hedging policy. Concerning insurance, its behavior would depend on the sign of

$\mu - (r + \lambda)L$. When this term is negative (large losses) the firm would not insure at all and would default whenever an accident occurs. In this case, shareholder value would be

$$V_{FB}(x) = x + \frac{\mu}{r + \lambda}, \quad (7)$$

since initial cash holdings x would be distributed immediately as a lump sum dividend, and shareholders would also receive a flow of dividends μdt until the first accident. The expected present value of this cash flow is $\frac{\mu}{r + \lambda}$. When $\mu - (r + \lambda)L$ is positive, $\frac{\mu}{r + \lambda}$ is smaller than $\frac{\mu - \lambda L}{r}$. In this case the firm would not keep any cash either but it would insure completely (or equivalently issue new equity after each accident) and would never default, leading to a different form of the shareholder value function:

$$V_{FB}(x) = x + \frac{\mu - \lambda L}{r}. \quad (8)$$

Thus we have established:

Proposition 1 *Consider the case where issuing new equity is costless ($f = 0$). The optimal policy of the firm is such that:*

- *the firm never holds any cash,*
- *the hedging policy is irrelevant,*
- *when $L > \frac{\mu}{r + \lambda}$ the firm does not buy insurance and defaults whenever an accident occurs:*

$$V_{FB}(x) = x + \frac{\mu}{r + \lambda};$$

- *when $L < \frac{\mu}{r + \lambda}$ the firm is indifferent between insuring completely or issuing new equity every time an accident occurs. The firm never defaults:*

$$V_{FB}(x) = x + \frac{\mu - \lambda L}{r}.$$

Note that when $f = 0$, the second friction ($r_0 < r$) does not matter since the firm never holds any cash. By contrast, when $f > 0$ and $r_0 = r$ (first friction only) shareholders

prefer to hoard cash without limits, in order to minimize the probability of having to issue new equity in the future. But then, no dividends are ever distributed, which means that the shareholder value maximization problem does not have a well defined solution.

The interesting case occurs when both $f > 0$ and $r_0 < r$. For tractability reasons we assume in the core of the text that $r_0 = 0$. The general case $r_0 > 0$ is examined in Section 4.2.

2.3 A Benchmark: Shareholder Value when Risk Management is Impossible

In order to be able to measure the gains from hedging and insurance, this section studies the case where these activities are impossible. The shareholder value function V is defined by formula (6), where issuance and dividend policies are optimized. Following Décamps et al. (2008) this section shows that the optimal financial policy of the firm depends on the magnitude of the issuance cost.

Before going further, we need to specify the mathematical formulation of the control problem faced by the firm. To do this, we introduce an operator M that models the impact of the issuance of new shares:

$$MV(x) = \max_{i \geq 0} \{V(x + i - f) - i\}.$$

Since the firm is free to issue new equity at any moment (for a fixed cost f) it must be that, for all x , $V(x) \geq MV(x)$, with equality for all the levels of x at which it is optimal to issue new equity. Similarly, since the firm can always distribute an amount z of dividends, to its shareholders (at no cost) its value function satisfies:

$$\forall x \quad \forall z \geq 0 \quad V(x) \geq V(x - z) + z.$$

Dividing by z and letting z converge to zero, we see that $V'(x) \geq 1$, with equality only in the region where dividends are paid.

External financing costs force the firm to retain some cash in order to reduce the probability of financial distress. This creates endogenous risk aversion for the firm's shareholders:

the shareholder value function V is concave¹¹ and the marginal value of cash holdings $V'(x)$ is decreasing in x . The concavity of the value function implies that the optimal dividend policy is always characterized by a target cash level x^* such that all cash is retained below x^* and distributed above it. This implies:

$$V(x) = x - x^* + V(x^*) \quad \text{for } x \geq x^*.$$

Moreover it can be proved that the optimal V is of class \mathcal{C}^2 , and thus that:¹²

$$V'(x^*) = 1, \quad V''(x^*) = 0. \quad (9)$$

The dividend policy has strong implications for the operator M . Indeed, whenever there is a new issue, the optimal amount is always¹³ $i = x^* - x + f$ so that $MV(x) = x + (V(x^*) - x^* - f)$. Since the firm cannot continue with a negative cash reserve (when the cash reserve is negative the only alternative is to issue new shares or liquidate the firm), the value function V must satisfy $V(x) = \max(0, MV(x))$ for $x \leq 0$. It is interesting to note that V is convex in the financial distress region. In particular, $V(0) = \max(0, V(x^*) - x^* - f)$. Finally, whenever $x > 0$, the firm's management always has the option to remain passive (no dividend, no issuance), in which case, the value function evolves like a discounted martingale, so that:

$$rV(x) = \mu V'(x) + \left(\frac{\sigma^2 + \sigma_R^2}{2} \right) V''(x) - \lambda [V(x) - V(x - L)].$$

Whenever remaining passive is not optimal it must be that:

$$rV(x) \geq \mu V'(x) + \left(\frac{\sigma^2 + \sigma_R^2}{2} \right) V''(x) - \lambda [V(x) - V(x - L)].$$

Therefore it is useful to introduce the second order differential operator with delay:

$$\mathcal{A}U(x) = \mu U'(x) + \left(\frac{\sigma^2 + \sigma_R^2}{2} \right) U''(x) - (r + \lambda)U(x) + \lambda U(x - L).$$

In all the paper, the characterization of the optimal policy of the firm will be obtained in three steps:

¹¹Concavity of the value function is a general property of this class of problems. see Clark and Kiessler (2002) for a general proof when the cash reserve process is continuous. Their proof does not extend to our framework, due to Poisson jumps.

¹²The second condition is often called super-contact (see Dumas (1991)).

¹³This is because there is no proportional cost of issuance. Introducing such a cost does not alter the qualitative properties of V . See Décamps et al. (2009) for details.

- showing that the value function satisfies some differential equation with endogenous boundary conditions (a free boundary problem),
- finding a feasible strategy that generates a value function that solves this problem,
- proving that no feasible strategy can strictly improve on this value function (verification theorem).

We are now in a position to characterize the value function when risk management is impossible.

We first consider the case where L is so large (the precise condition will be given below) that the firm always defaults after an accident: $V(x - L) \equiv 0$ for all $x \leq x^*$. In this case the operator \mathcal{A} takes a simpler expression:

$$\mathcal{A}U(x) = \left(\frac{\sigma^2 + \sigma_R^2}{2} \right) U''(x) + \mu U'(x) - (r + \lambda)U(x).$$

Moreover, if the issuance cost is so large (here also the precise condition will be given below) that shareholders prefer to default whenever the firm runs out of cash ($V(x) = 0$ for every $x \leq 0$), the previous discussion suggests that the optimal shareholder value function V satisfies:

$$\mathcal{A}V(x) = 0 \quad \text{on some interval } [0, x^*]$$

with the boundary conditions

$$\forall x \leq 0, V(x) = 0, \quad V'(x^*) = 1 \text{ and } V''(x^*) = 0.$$

It turns out that Jeanblanc and Shiryaev (1995), have established that there is a unique pair (V_0, x_0^*) solution to this boundary problem:

$$\begin{cases} \left(\frac{\sigma^2 + \sigma_R^2}{2} \right) V_0'' + \mu V_0' - (r + \lambda)V_0 = 0 \text{ for } 0 < x \leq x_0^* \\ V_0(0) = 0, V_0'(x_0^*) = 1 \text{ and } V_0''(x_0^*) = 0 \end{cases}$$

The next proposition states that the value function V coincides indeed with V_0 for large losses L and large issuance cost f . More precisely, we have:

Proposition 2 *In the case of large potential losses and large issuance cost (specifically, when $L \geq x_0^*$ and $f \geq \frac{\mu}{r+\lambda} - x_0^*$) we have $V = V_0$. Consequently, optimal liquidity management is such that:*

- *The firm retains all cash up to some threshold x_0^* and distributes all cash above x_0^* .*
- *The firm never issues new equity and defaults whenever $x < 0$.*

Proof: See the Appendix. ◇

A closed form solution is available by introducing $\theta_1 < 0 < \theta_2$, the roots of the quadratic equation

$$r + \lambda = \mu\theta + \frac{1}{2}(\sigma^2 + \sigma_R^2)\theta^2.$$

The shareholder value function is given by:

$$V_0(x) = \frac{e^{\theta_2 x} - e^{\theta_1 x}}{\theta_2 e^{\theta_2 x_0^*} - \theta_1 e^{\theta_1 x_0^*}} \text{ for } x \leq x_0^* \quad (10)$$

$$= x - x_0^* + \frac{\mu}{r + \lambda} \text{ for } x > x_0^*. \quad (11)$$

where the target cash level for dividend distribution is:

$$x_0^* = \frac{1}{\theta_2 - \theta_1} \ln \frac{\theta_1^2}{\theta_2^2}. \quad (12)$$

This case is illustrated in Figure 1.

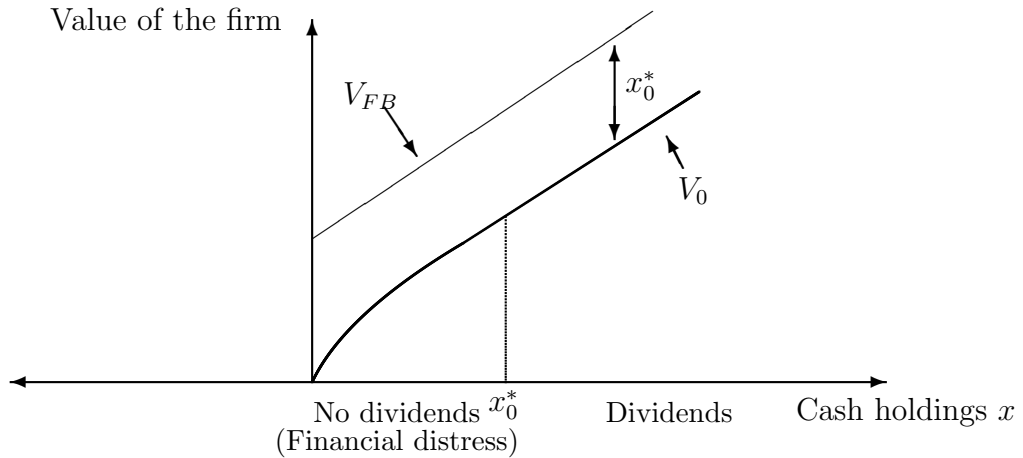


Figure 1: Shareholder value $V_0(x)$ compared with the first best value $V_{FB}(x)$.

In our simple set-up, the target cash level x_0^* is a good measure of the cost of financial frictions: indeed for x large (specifically for $x \geq x_0^*$, i.e. outside the financial distress region) the difference between $V_{FB}(x)$ and $V_0(x)$ is precisely equal to x_0^* . Thus the cost of financial frictions is measured exactly by the amount of cash reserves that have to be kept idle by the firm (remember our assumption that cash reserves are not remunerated). In the more general case where $r_0 > 0$ (studied in Section 4.1) the target cash level is $x^* > x_0^*$, and the cost of financial frictions is measured by $(1 - \frac{r_0}{r})x^*$, the present value of foregone interest $(r - r_0)x^*$ on the target cash reserve.

The same reasoning can be employed in the case where L is large but f is small. Indeed, assume for a while that there is a unique pair $(V_1, x_1^*(f))$ that solves the following free boundary problem:

$$\left(\frac{\sigma^2 + \sigma_R^2}{2}\right) V_1'' + \mu V_1' - (r + \lambda)V_1 = 0 \text{ for } x \leq x_1^* \quad (13a)$$

$$V_1'(x_1^*) = 1 \text{ and } V_1''(x_1^*) = 0 \quad (13b)$$

$$V_1(x) = \max(0, x + V_1(x_1^*) - x_1^* - f) \text{ for } x \leq 0 \quad (13c)$$

The next proposition states that the value function V coincides with V_1 for large losses L and small issuance cost f . More precisely, we have:

Proposition 3 *In the case of large potential losses and small issuance costs (specifically when $L \geq x_0^*$ and $f < \frac{\mu}{r+\lambda}x_0^*$), we have $V = V_1$. Consequently, optimal cash management is characterized by:*

- *The firm retains all cash up to some threshold $x_1^*(f)$ (which is smaller than x_0^*) and distributes all cash above $x_1^*(f)$.*
- *Whenever the firm runs out of cash ($x = 0$), it issues new equity for an amount $x_1^*(f) + f$.*
- *The firm defaults if and only if there is an accident.*

Proof: See the Appendix. ◇

It is easy to check that x_1^* is an increasing function of f such that $x_1^*(0) = 0$ and $x_1^*(f) = x_0^*$ for $f \geq \frac{\mu}{r+\lambda}$. Moreover, $V_1(x)$ converges to $V_{FB}(x) = x + \frac{\mu}{r+\lambda}$ when f converges to zero. Here again, $x_1^*(f)$ is a good measure of the cost of financial frictions.

A comparison of Propositions 2 and 3 shows the impact of the issuance cost on the dividend policy and the default probability of the firm when risk management is impossible. When the issuance cost is high (Proposition 2) the firm has to reach a high level of cash x_0^* before distributing dividends. Moreover the probability of default is high since this default can be provoked either by an accident or by a sufficiently long stretch of negative cash flows that exhausts the firm's cash reserves ($x = 0$). By contrast, when the issuance cost f is smaller (Proposition 3) the target level of cash $x_1^*(f)$ is lower ($x_1^*(f) < x_0^*$) and default can only be provoked by an accident, since the firm always issues new equity when it runs out of cash. As a result, the firm's risk aversion is reduced, dividends are distributed more often and the probability of default is lower.

In order to study the impact of hedging and insurance on shareholder value, we now consider the case where risk management is possible.

3 Optimal Risk Management

We now assume that the firm has access to perfect risk management instruments: it can hedge the Brownian risk at no cost and insure potential accidents at actuarial premiums. The dynamics of the cash reserves is now given by

$$dX_t^u = [(\mu + r_0 X_t - \lambda L i_t)dt + \sigma dW_t] + [(1 - h_t)\sigma_R dW_t^R - (1 - i_t)LdP_t] - dZ_t + dI_t - dF_t, \quad (14)$$

where the control variables of the firm are represented by an adapted process $u_t = (I_t, Z_t, h_t, i_t)$ with I_t is the cumulative issuance process defined as in condition (2), F_t is the cumulative issuance cost process defined as (3), and Z_t is the cumulative dividend

process. Moreover $h_t \in [0, 1]$ represents the fraction of the Brownian risk that is hedged and $i_t \in [0, 1]$ the fraction of the Poisson loss that is insured. The optimal shareholder value function is found by maximizing the expected discounted value of dividends up to default time τ_B , the first time the controlled cash flow process X_t^u falls below zero. Formally

$$V(x) = \max_{I, Z, h, i} E_x \left[\int_0^{\tau_B} e^{-rt} (dZ_t - dI_t) \right]. \quad (15)$$

3.1 Some Properties of the Solution

As in the benchmark case, the optimal choices of I and Z are characterized respectively by the operator M defined as:

$$MV(x) = \max_{i \geq 0} \{V(x + i - f) - i\}$$

and by the first derivative of V . Again, costly external financing induces the concavity of V and thus the optimal dividend policy will be again characterized by a target cash level x^* such that all cash is retained below x^* and distributed above it. Because the marginal value of cash for shareholders is non-increasing, this target cash level is simply $x^* = \inf\{x, V'(x) = 1\}$. Moreover, optimality implies that V is of class \mathcal{C}^2 , and thus that:

$$V'(x^*) = 1, \quad V''(x^*) = 0. \quad (16)$$

As in the benchmark case, it is easily seen that

$$\forall x \geq 0 \quad \forall i \geq 0 \quad V(x) \geq V(x + i) - i - f. \quad (17)$$

Since V is concave and $V'(x^*) = 1$ we have

$$\forall x < x^* : \max_{i \geq 0} \{V(x + i) - i - f\} = V(x^*) - x^* - f + x.$$

Therefore (17) is equivalent to

$$\forall x \geq 0 \quad V(x) - x \geq V(x^*) - x^* - f$$

Since $V'(x) \geq 1$, we have that

$$\min_{x \geq 0} [V(x) - x] = V(0),$$

and therefore $V(0) \geq V(x^*) - x^* - f$, with equality if and only if the firm issues new equity when it runs out of cash. Alternatively, if it never issues new equity: it is liquidated when $x \leq 0$ and then $V(0) = 0$. As a consequence, $V(x) = \max(0, x + V(x^*) - x^* - f)$ for $x \leq 0$. Therefore, V is locally convex in this region.

In our model, when a firm is in financial distress with a low level of cash, it may raise new funds. However, we will show that it is never optimal to issue new equity when the cash reserves are strictly positive. The intuition is as follows. Because of the fixed issuance cost f , it is always better to defer external financing as long as the firm's cash reserve is zero.

• Optimal risk management policy

On the interval $(0, x^*)$, there is no equity issuance and no dividend distribution ($dI_t = dZ_t = 0$). Therefore the cash reserve dynamics is given by:

$$dX_t = (\mu - \lambda Li_t)dt + \sigma dW_t + (1 - h_t)\sigma_R dW_t^R - L(1 - i_t)dP_t.$$

Thus, the optimal value function V satisfies the Hamilton-Jacobi-Bellman equation on $(0, x^*)$:

$$rV(x) = \max_{i,h} (\mu - \lambda Li)V'(x) + \frac{1}{2} (\sigma^2 + (1 - h)^2 \sigma_R^2) V''(x) - \lambda [V(x) - V(x - (1 - i)L)]. \quad (18)$$

The concavity of V on $(0, x^*)$ gives rise to a demand for hedging. Since hedging is costless, it is even optimal to hedge fully:

$$\forall x \in (0, x^*) \quad h^*(x) \equiv 1.$$

The optimal choice of insurance is more subtle. Even if V is concave on $(0, x^*)$ full insurance is not optimal when an accident sends the firm in the region $x < 0$ where the value function V is convex. This creates an incentive to gamble by not purchasing insurance contracts, even if they are fairly priced.

The marginal gain from partial insurance is measured by $\lambda V(x - (1 - i)L)$ while the marginal cost is $i\lambda LV'(x)$. Thus the optimal insurance decision i^* satisfies the condition

$$\lambda V(x - (1 - i^*)L) - i^* \lambda LV'(x) = \max_i [\lambda V(x - (1 - i)L) - i \lambda LV'(x)].$$

In the region where the function between brackets is concave in i , its maximum is obtained for $i = 1$. In the region where it is convex, its maximum is either 0 or 1. Thus we can assume without loss of generality that $i^* = 1$ or $i^* = 0$. In order to measure the gain from insurance, we define the operator $D(i)$ as

$$D(i)U(x) = \frac{\sigma^2}{2}U''(x) + (\mu - \lambda Li)U'(x) - (r + \lambda)U(x) + \lambda U(x - (1 - i)L).$$

Note that $i^*(x) = 0$ if and only if $D(0)V(x) \geq D(1)V(x)$, otherwise $i^*(x) = 1$.

3.2 Optimal Risk Management when Issuing Costs are High

When issuing costs are high, we establish below that the optimal insurance decision is given by:

$$i^*(x) = 0 \quad \text{if} \quad 0 \leq x < \bar{x}_0 \tag{19}$$

$$= 1 \quad \text{if} \quad \bar{x}_0 \leq x \leq \bar{x}_1, \tag{20}$$

where \bar{x}_0 and \bar{x}_1 are respectively the insurance and the dividend thresholds. In other words, buying insurance is optimal when the firm has enough cash ($x \in [\bar{x}_0, \bar{x}_1]$). The intuition is given by the shape of the value function: convex for x negative, concave anywhere else. Since hedging eliminates small risks in the concavity region, it is always optimal to hedge fully (at least when hedging is costless). By contrast, insurance is intended to cover large risks, that may make X_t jump below 0, in the convexity region. This is why insurance is not optimal when the firm is cash-poor.

As explained above, the shareholder value function V can be obtained in three steps: first finding a \mathcal{C}^2 solution $(\bar{V}, \bar{x}_0, \bar{x}_1)$ to the following free boundary problem (see Lemma 2 in the Appendix):

$$\begin{cases} D(0)\bar{V}(x) = 0 & 0 < x < \bar{x}_0 \\ D(1)\bar{V}(x) = 0 & \bar{x}_0 < x < \bar{x}_1 \\ \bar{V}(x) = 0, & x \leq 0 \\ \bar{V}'(\bar{x}_1) = 1, \bar{V}''(\bar{x}_1) = 0. \end{cases} \tag{21}$$

Secondly, we have to check that the smooth solution \bar{V} of the free boundary problem (21) is attainable and finally that it satisfies the appropriate verification theorem (Theorem 2

in the appendix).

The next proposition characterizes the value function and the optimal insurance policy when the loss is not too large.

Proposition 4 *Let $(\bar{V}, \bar{x}_0, \bar{x}_1)$ be the unique solution of the free boundary problem (21) and assume that $f \geq \frac{\mu - \lambda L}{r} - \bar{x}_1$ and $L \leq \frac{\mu}{r + \lambda}$.*

The optimal value function V coincides with \bar{V} . It is characterized by three regimes:

- $0 \leq x \leq \bar{x}_0$ (no insurance regime):
- $\bar{x}_0 \leq x \leq \bar{x}_1$ (insurance regime):
- $x \geq \bar{x}_1$ (dividend regime):

Proof: See the Appendix. ◇

Proposition 4 shows that insurance is bought when the firm is cash-rich ($x > \bar{x}_0$) and losses are not too high ($L \leq \frac{\mu}{r + \lambda}$). The properties of the value function corresponding to optimal policy when $L \leq \frac{\mu}{r + \lambda}$ are summarized in the following figure:

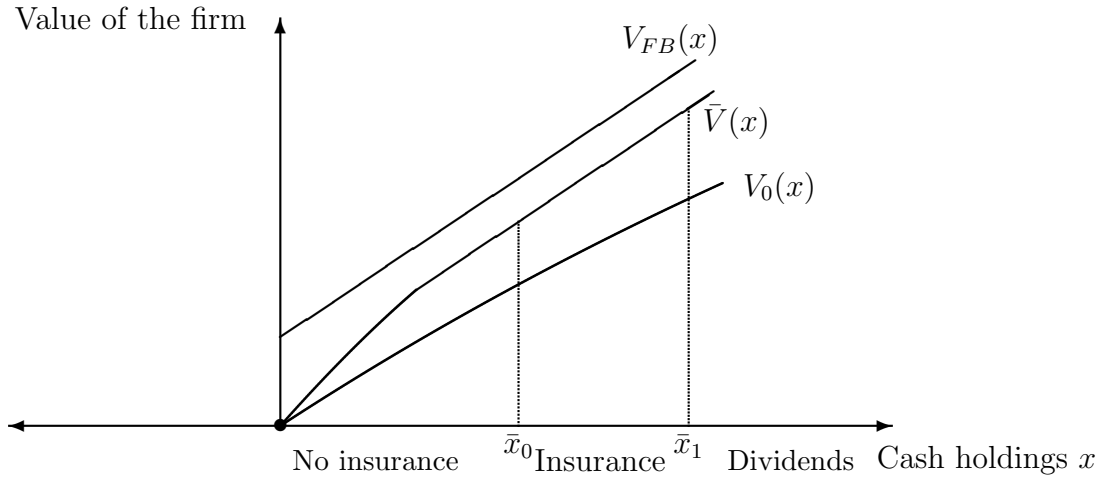


Figure 2: The gains from insurance: The value $\bar{V}(x)$ of the firm that insures optimally compared with the value $V_0(x)$ of the firm that does not insure, and the first best value $V_{FB}(x)$.

The gains from insurance can be measured by the difference between the cost of financial frictions without and with insurance. The same is true for hedging.

Note that the no-insurance regime would remain even if the firm had unlimited liability. This result seems counterintuitive since the convex kink in the firm's value function in zero is usually associated with the limited liability option. However, it is interesting to note that the convex kink remains when the firm has unlimited liability. Suppose indeed shareholders are forced to inject capital every time x becomes negative. In that case, V is linear with slope one on $] - \infty, 0]$ since we have

$$V(x) = x + V(\bar{x}_1) - \bar{x}_1 - f.$$

With unlimited liability, the marginal value of cash is still greater than one in a right neighborhood of zero due to the cost of external funding. As a consequence, there is still a convex kink at zero which implies that insurance is not profitable for the firm when x is close to zero. Formally, the expected gain from full insurance is $\lambda(V(0) - V(-L)) = \lambda L$ while the expected cost is $\lambda L V'(0^+)$ which is higher.

For completeness, let us examine the case of large losses ($L > \frac{\mu}{r+\lambda}$). We show that in that case, the firm should not insure at all (i.e., $i^* \equiv 0$). The optimal value function when issuance costs are high has the same form as in the benchmark case. In particular the optimal dividend policy is to pay out any excess cash above a threshold \tilde{x}_0 which can be computed explicitly (see Equation (A.10) in the Appendix). As a consequence it is not optimal for shareholders to insure large risks when external financing is costly, even if insurance is fairly priced.

Indeed, the next proposition characterizes the value function V for large potential losses L and large issuance cost f . More precisely, we have:

Proposition 5 *When $L \geq \frac{\mu}{r+\lambda}$ and $f \geq \frac{\mu}{r+\lambda} - \tilde{x}_0$, the optimal strategy is never to insure ($i^* \equiv 0$) and the value function is given by Equation (A.9). Consequently:*

- *The firm retains all cash up the threshold \tilde{x}_0 and distributes all cash above \tilde{x}_0 .*
- *The firm never issues new equity.*

- *The firm does not buy any insurance.*
- *The firm defaults if either there is an accident or it runs out of cash.*

Proof: See the Appendix. ◇

4 Robustness Checks

4.1 Small Issuance Cost

When the issuance cost is small, the solution is qualitatively similar but more complex to characterize. Nevertheless, when losses are large the characterization of the shareholder value function follows from Proposition 5. The optimal pattern of insurance is still that cash poor firms find do not insure because an accident brings them in the region where the value function is convex. When losses are large, the shareholders are effectively poorer because they face a higher refinancing risk and thus prefer to pay themselves some dividend rather than buying insurance. We have,

Proposition 6 *Assume that $L \geq \frac{\mu}{r+\lambda}$ and $f < \frac{\mu}{r+\lambda} - \tilde{x}_0$. Then, the value function is given by Equation (A.11) in the Appendix. Consequently,*

- *The firm retains all cash up to some threshold \tilde{x}_1 and distributes all cash above \tilde{x}_1 .*
- *Whenever the firm runs out of cash ($x = 0$) it issues new equity for an amount $\tilde{x}_1 + f$.*
- *The firm does not buy any insurance.*
- *The firm defaults if and only if there is an accident.*

Proof: See the Appendix. ◇

4.2 Remuneration of Cash Holdings

In this section, we assume that cash reserves are remunerated, that is $r_0 > 0$. Our focus is on the impact on risk management policies when the issuance cost f is large. We prove that the results of Proposition 4 are robust. More precisely, the optimal policy is characterized by complete hedging $h^*(x) \equiv 1$ and a bang-bang insurance strategy depending on the level of cash reserves, similar to that of Proposition 4.

Proposition 7 *The optimal value function V is characterized by three regimes:*

- $0 \leq x \leq x_0$ (no insurance regime):
- $x_0 \leq x \leq x_1$ (insurance regime):
- $x \geq x_1$ (dividend regime):

where thresholds x_0, x_1 are characterized by Problem (A.12) in the Appendix.

4.3 Costly Hedging

We assume in this section that π_h , the cost of hedging is positive, but not too large, and by contrast that the cost of issuing new equity is large. In this case, the dynamics of cash reserves is given by

$$dX_t = (\mu + r_0 X_t - \frac{\sigma_R^2}{2} \pi_h h_t - \lambda L i_t) dt + (1 - h_t) \sigma_R dW_t^R - L(1 - i_t) dP_t - dZ_t.$$

As in the previous section, we can associate to this dynamics of cash reserves the linear operator given by

$$A(h, i)U(x) = \frac{1}{2}(\sigma^2 + (1-h)^2 \sigma_R^2) f''(x) + (\mu + r_0 x - \frac{\sigma_R^2}{2} \pi_h h - \lambda L i) f'(x) - r f - \lambda(f(x) - f(x-L))$$

that characterizes the optimality of hedging/insurance decisions.

Proposition 8 *If $\pi_h < \frac{\mu \sigma_R^2}{\sigma^2 + \sigma_R^2}$ then there exists a concave twice differentiable solution $(\hat{V}, \hat{x}_0, \hat{x}_1)$ of the free boundary problem*

$$\begin{cases} A(1, 0)\hat{V}(x) = 0 \text{ for } 0 \leq x \leq \hat{x}_0 \\ A(0, 0)\hat{V}(x) = 0 \text{ for } \hat{x}_0 \leq x \leq \hat{x}_1 \\ \hat{V}(0) = 0, \hat{V}'(x_1) = 1, \hat{V}''(\hat{x}_1) = 0. \end{cases}$$

Proof: See the appendix. ◇

This allows to characterize the shareholder value function when hedging costs are positive but not too large.

Proposition 9 *If $\pi_h < \frac{\mu\sigma_R^2}{\sigma^2 + \sigma_R^2}$ and $L \geq \frac{\mu + r_0\hat{x}_1}{r + \lambda}$ (where \hat{x}_1 is given by Proposition 8) the shareholder value function coincides with \hat{V} . Therefore the optimal solution is such that the firm only hedges when cash reserves are low:*

$$h(x) = \mathbb{I}_{x < \hat{x}_0}.$$

Proof: See the Appendix. ◇

It is interesting to notice the completely opposed impacts of cash holdings on insurance and hedging. Proposition 9 establishes that cash rich firms should not hedge when hedging is costly ($\pi_h > 0$). By contrast, Proposition 4 shows that cash poor firms should not insure (of course from the perspective of shareholders, not that of society as a whole).

5 Estimating the Gains from Risk Management

In our model, risk management allows to reduce the cost of financial frictions. The cost of these frictions can be proxied by the target level of cash that must be attained before distributing dividends. Consider for example the gains from hedging, in a situation where insurance is not available. When hedging is costless, the firm will always hedge fully. This comes from the concavity of the shareholder value function in the region $x > 0$. Then if issuing costs are large, the gains from hedging can be immediately assessed from Proposition 2, by looking at how much x_0^* is reduced when volatility is reduced from $\sigma^2 + \sigma_R^2$ to σ^2 .

Let us recall the formula (12) giving x_0^* :

$$x_0^* = \frac{1}{\theta_2 - \theta_1} \ln \frac{\theta_1^2}{\theta_2^2}$$

where $\theta_1 < 0 < \theta_2$ are the roots of the quadratic equation

$$r + \lambda = \mu\theta + \frac{1}{2}(\sigma^2 + \sigma_R^2)\theta^2.$$

When hedging is costless (and insurance is impossible or too costly) the target cash level is given by the same formulas, the only difference being that $\sigma^2 + \sigma_R^2$ is replaced by σ^2 . It is therefore legitimate to define a function $x^*(\mu, \sigma^2)$ by the two conditions above. The next proposition shows how x_0^* varies with μ and σ^2 .

Proposition 10 : *The cost of financial frictions x_0^* is a single peaked function of μ , and an increasing function of σ^2 .*

The following figures represent the cost of financial frictions x_0^* as a function of μ and σ^2 . Note that x^* is bounded above by $\frac{\mu}{r}$.

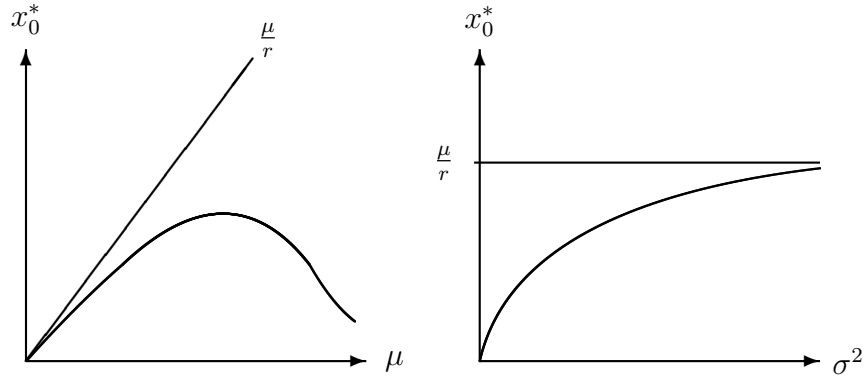


Figure 3: The cost of financial frictions as a function of μ and σ^2 .

Maybe the most striking of these properties is the non-monotonicity of x_0^* with respect to μ , which measures the net profitability of the firm. Highly profitable firms are not really affected by financial frictions because their probability of financial distress is small. Conversely, barely profitable firms have little to lose from failure. It is the intermediate firms that are hurt the most by the risk of default.

6 Conclusion: Who Should Hedge?

This paper has derived the optimal hedging and insurance policies of a firm that faces cash flow risks. We have found that these policies are more complex than expected. In particular they exhibit non linearities and even non monotonic behavior with respect to variables of interest, such as liquidity and profitability. Further empirical work on this topic should therefore adopt specifications that account for these non linearities.

We can also derive from our model several testable implications about which firms are more likely to hedge,¹⁴ in the hope to shed light on the mixed findings of the empirical literature. We found in Proposition 9 that, provided a firm has decided to use hedging instruments optimally, it will tend to buy hedging ($h = 1$) when it is cash-poor ($x \leq \hat{x}_0$) and not hedge ($h = 0$) when it is cash-rich ($x > \hat{x}_0$). Consider now the prior decision to create, within the firm, a risk management unit and to hire the personnel able to manage the hedging position of the firm according to the instructions given by top management. This decision is optimal if the gains from hedging exceed the cost of creating this risk management unit.

An empirical economist who has collected panel data on the balance sheets of a given population of firms could estimate the parameters of our model such as expected profitability μ , and volatility of earnings σ^2 . Our model predicts that the probability that a firm creates a risk management unit (or participates in the derivatives market) is an increasing function of the gain from hedging, measured by the reduction in the costs of financial frictions obtained by hedging. When insurance is not available we saw that this gain could be measured by:

$$G_0 = x^*(\mu, \sigma^2 + \sigma_R^2) - x^*(\mu, \sigma^2). \quad (22)$$

We already saw that x^* was an increasing, concave, function of σ^2 . Thus we deduce immediately from formulas (12) and (22) that:

$$\frac{\partial G_0}{\partial \sigma_R^2} > 0 \quad \text{and} \quad \frac{\partial G_0}{\partial \sigma^2} < 0.$$

¹⁴For simplicity, we focus on the hedging decision, since the formulas for the gains from insurance are more complex.

This means that the gain from hedging increases with the volatility σ_R^2 of the hedgeable risk and decreases with the volatility σ^2 of the “operating” risk. More interestingly, the impact of μ is non monotonic, as illustrated by Figure 4:

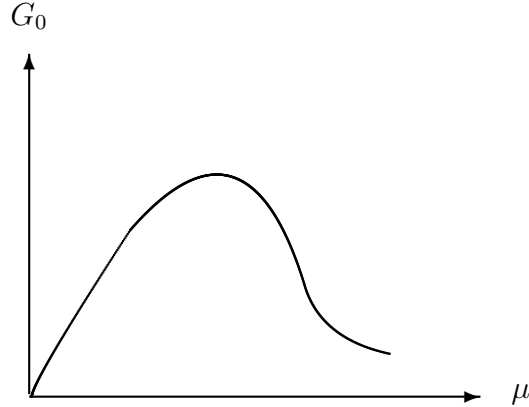


Figure 4: The gain from hedging as a function of profitability μ .

Thus profitability has a non monotonic (and highly non linear) impact on the gains from hedging. This may explain why empirical studies that use linear specifications have failed to derive any significant impact of profitability on the likelihood that a firm decides to hedge.

Appendix A

We first recall the following result in order to apply the verification procedure.

Theorem 1 *Assume that a function U satisfies the following quasi-variational inequalities on $(0, \infty)$*

$$\begin{aligned}\mathcal{A}U(x) &\leq 0 \\ U(x) &\geq MU(x) \\ U'(x) &\geq 1\end{aligned}$$

and

$$U(x) \geq \max(0, MU(x)) \quad \text{for } x \leq 0.$$

then $U \geq V$.

Proof of Theorem 1: The proof is an immediate adaptation of Oksendal and Sulem (2005), Theorem 5.2 part a) page 75.

Proof of Proposition 2: Using Theorem 1, it is enough to check that for every $x \geq 0$,

$$\mathcal{A}V_0(x) \leq 0,$$

and

$$V_0(x) \geq MV_0(x) = x + \frac{\mu}{r + \lambda} - x_0^* - f.$$

This will establish that $V_0 \geq V$. Because $V_0(x) - x \geq 0$ for every $x \geq 0$ and $\frac{\mu}{r + \lambda} - x_0^* - f \leq 0$ by assumption, it is thus straightforward that $V_0 \geq MV_0$.

Since $x_0^* \leq L$ and $V_0(x) = 0$ for $x \leq 0$, we have $\mathcal{A}V_0(x) = 0$ for $0 \leq x \leq x_0^*$. Now, for $x \geq x_0^*$, we have

$$\mathcal{A}V_0(x) = -(r + \lambda)(x - x_0^*) + \lambda V_0(x - L).$$

This is clearly negative when $x_0^* \leq x < L$ (this is because $V_0(x - L)$ is then equal to 0).

But, for any $x \geq L$, one has $V_0(x - L) \leq x - L - x_0^* + \frac{\mu}{r + \lambda}$. Thus,

$$\mathcal{A}V_0(x) \leq (r + \lambda)(x - x_0^*) + \lambda \left(x - L - x_0^* + \frac{\mu}{r + \lambda} \right) = -r(x - x_0^*) + \lambda \left(\frac{\mu}{r + \lambda} - L \right) < 0.$$

This ends the proof that $V_0 \geq V$. Finally, the value function V_0 is attainable (since it corresponds to the admissible policy: *pay dividends whenever cash reserves exceed x_0^** , the converse inequality ($V \geq V_0$) comes directly from the definition of the value function V .

Proceeding as in the proof of Proposition 2, we now establish Proposition 3.

Proof of Proposition 3: We first prove that $\mathcal{A}V_1(x) = 0$ for $0 \leq x \leq x_1^*$. To see this, note first that $V_1(x - L) = 0$ for every $0 \leq x \leq x_1^*$. Indeed, since V_1 is concave, one has

$$V_1(0) \leq V_1(x_1^*) - x_1^* V_1'(x_1^*).$$

Since $V_1(0) \geq 0$ and $V_1'(x_1^*) = 1$, this implies:

$$x_1^* \leq V_1(x_1^*) = \frac{\mu}{r + \lambda} \leq L.$$

Thus, $x - L \leq 0$ for every $0 \leq x \leq x_1^*$ and

$$\begin{aligned} x - L &\leq x_1^* - \frac{\mu}{r + \lambda} \\ &= -f - V_1(0). \end{aligned}$$

This implies

$$x - L < -V_1(0).$$

Therefore, $V_1(x - L) = \max(0, x - L + V_1(0)) = 0$ for every $0 \leq x \leq x_1^*$, which implies $\mathcal{A}V_1(x) = 0$ for $0 \leq x \leq x_1^*$. Now, for $x \geq x_1^*$, we have

$$\mathcal{A}V_1(x) = -(r + \lambda)(x - x_0^*) + \lambda V_1(x - L).$$

But, for every $x \geq 0$, one has $V_0(x) \leq x - x_1^* + \frac{\mu}{r + \lambda}$. Thus,

$$\mathcal{A}V_1(x) \leq -r(x - x_1^*) + \lambda\left(\frac{\mu}{r + \lambda} - L\right) < 0.$$

The fact that $V_1(x) \geq MV_1(x) = x + V_1(0)$ for every $x \geq 0$ results directly from the fact that V_1 is concave on $(0, \infty)$ and $V_1' \geq 1$. This establishes that $V_1 \geq V$.

Finally the value function V_1 corresponds to an admissible policy. The reverse inequality ($V \geq V_1$) comes from the definition of the value function V . It remains to prove that the free boundary problem (13a)-(13b)-(13c) admits a unique solution. For a fixed x_1^* , there is a unique solution V_1 of (13a) that satisfies (13b). It is given by:

$$V_1(x) = \frac{\theta_2^2 e^{\theta_1(x - x_1^*)} - \theta_1^2 e^{\theta_2(x - x_1^*)}}{\theta_1 \theta_2 (\theta_2 - \theta_1)}, x < x_1^*, \quad (\text{A.1})$$

$$V_1(x) = x - x_1^* + \frac{\mu}{r + \lambda}, x > x_1^*. \quad (\text{A.2})$$

To satisfy (13c), the target cash level $x_1^*(f)$ must be such that:

$$V_1(0) = V_1(x_1^*) - x_1^* - f \geq 0,$$

which is equivalent to two conditions:

$$x_1^* + \frac{\theta_2^2 e^{-\theta_1 x_1^*} - \theta_1^2 e^{-\theta_2 x_1^*}}{\theta_1 \theta_2 (\theta_2 - \theta_1)} = \frac{\mu}{r + \lambda} - f, \quad (\text{A.3})$$

and

$$\frac{\mu}{r + \lambda} - x_1^* - f \geq 0. \quad (\text{A.4})$$

Therefore, the existence of a unique pair (V_1, x_1^*) solution of (8) is guaranteed by

Lemma 1 *There is a unique solution $x_1^*(f)$ to Equation (A.3). Moreover if $f \leq \frac{\mu}{r + \lambda} - x_0^*$, it satisfies condition (A.4).*

Proof of Lemma 1: Define an auxiliary function:

$$\varphi(x) = x + \frac{\theta_2^2 e^{-\theta_1 x} - \theta_1^2 e^{-\theta_2 x}}{\theta_1 \theta_2 (\theta_2 - \theta_1)} + f - \frac{\mu}{r + \lambda}.$$

It is straightforward to see that φ is a decreasing¹⁵ function with $\varphi(0) = f > 0$ and $\varphi(+\infty) = -\infty$. Therefore there is a unique x_1^* such that $\varphi(x_1^*) = 0$. Moreover (12) implies that $\varphi(x_0^*) = x_0^* + f - \frac{\mu}{r+\lambda}$ which is negative. Thus $x_1^* < x_0^*$ and $x_1^* + f - \frac{\mu}{r+\lambda} \leq 0$. \blacksquare

The proofs of the optimality of insurance policies rely on the following verification theorem.

Theorem 2 *Consider any function U that satisfies the following quasi-variational inequalities on $(0, \infty)$*

$$\max_{i \in \{0,1\}} D(i)U(x) \leq 0,$$

$$U(x) \geq MU(x),$$

$$U'(x) \geq 1,$$

and

$$U(x) \geq \max(0, MU(0)) \quad \text{for } x \leq 0.$$

then $U \geq V$.

Proof of Theorem 2: It is an easy adaptation of Øksendal and Sulem (2005) Theorem 6.2 part a) page 83.

Proof of Proposition 4:

At this stage, we need to introduce some notation. By analogy with Section 2, let us denote by $\gamma_1 < 0 < \gamma_2$ the roots of the characteristic equation corresponding to $i = 1$ (insurance):

$$(\mu - \lambda L)\gamma + \frac{1}{2}\sigma^2\gamma^2 = r, \quad (\text{A.5})$$

and by $\bar{\theta}_1 < 0 < \bar{\theta}_2$ the roots of the characteristic equation corresponding to $i = 0$ (no insurance):¹⁶

$$\mu\theta + \frac{1}{2}\sigma^2\theta^2 = r + \lambda. \quad (\text{A.6})$$

Next proposition shows that there is a solution $(\bar{V}, \bar{x}_0, \bar{x}_1)$ to (21) such that $\bar{x}_0 \leq L$.

Lemma 2 *Assume $L \leq \frac{\mu}{r+\lambda}$. There is a unique C^2 solution $(\bar{V}, \bar{x}_0, \bar{x}_1)$ to (21). \bar{V} is concave and is given in closed form*

$$\bar{V}(x) = \begin{cases} \bar{A}(e^{\bar{\theta}_2 x} - e^{\bar{\theta}_1 x}) & \text{for } x \leq \bar{x}_0 \\ \bar{B}e^{\gamma_1 x} + \bar{C}e^{\gamma_2 x} & \text{for } \bar{x}_0 \leq x \leq \bar{x}_1 \\ x + \frac{\mu - \lambda L}{r} - \bar{x}_1 & \text{for } x \geq \bar{x}_1, \end{cases}$$

where \bar{A} , \bar{B} , \bar{C} are explicit functions of the parameters. The thresholds \bar{x}_0 and \bar{x}_1 are given by:

$$\bar{x}_0 = \frac{1}{\bar{\theta}_2 - \bar{\theta}_1} \ln \left(\frac{1 - L\bar{\theta}_1}{1 - L\bar{\theta}_2} \right). \quad (\text{A.7})$$

$$\bar{x}_1 = \bar{x}_0 + \frac{1}{\gamma_2 - \gamma_1} \ln \left(\frac{\gamma_1^2(1 - L\gamma_2)}{\gamma_2^2(1 - L\gamma_1)} \right). \quad (\text{A.8})$$

¹⁵To see this, note that φ' is decreasing (since $\varphi''(x) = \frac{\theta_2\theta_1}{\theta_2 - \theta_1}(e^{-\theta_1 x} - e^{-\theta_2 x}) < 0$) and $\varphi'(0) = 0$.

¹⁶This assumes implicitly that $\bar{x}_0 \leq L$, so that $V(x - L) = 0$ in the no-insurance region. This will be checked in Proposition 2.

Proof: The resolution of the free boundary problem (21) yielding to Equations (A.7) and (A.8) is straightforward. Note that formula (A.8) shows that:

$$\bar{x}_1 > \bar{x}_0 \Leftrightarrow \gamma_1^2(1 - L\gamma_2) > \gamma_2^2(1 - \gamma_2).$$

Since $\gamma_1 < 0 < \gamma_2$ this is equivalent to

$$L < \frac{\gamma_1 + \gamma_2}{\gamma_1\gamma_2} = \frac{\mu - \lambda L}{r},$$

which is guaranteed by our assumption $\mu > (r + \lambda)L$. This assumption also implies that $\bar{\theta}_2 < \frac{1}{L}$ and $\gamma_2 < \frac{1}{L}$, so that formulas (A.7) and (A.8) are well defined. We only check the first condition, the second being similar. $\bar{\theta}_2$ is by definition the largest root of $\psi(\theta) = \mu\theta + \frac{\sigma^2\theta^2}{2} - (r + \lambda)$. It is immediate that $\mu > (r + \lambda)L$ implies that $\psi(\frac{1}{L}) > 0$ and thus that $\bar{\theta}_2 < 1/L$.

The last thing that has to be checked is that the threshold \bar{x}_0 is lower than L . Given Equation (A.7), this is equivalent to show that

$$\ln(1 - L\bar{\theta}_1) - (1 - L\bar{\theta}_1) \leq \ln(1 - L\bar{\theta}_2) - (1 - L\bar{\theta}_2).$$

Now, the assumption on L implies that $\bar{\theta}_1 < 0 < \bar{\theta}_2 \leq \frac{1}{L}$. We conclude by remarking that the function

$$F(x) = \ln(1 - Lx) - (1 - Lx)$$

is increasing on $(-\infty, \frac{1}{L})$. ◇

Coming back to the proof of main Proposition, we first check that the solution \bar{V} to the free boundary problem (21) satisfies the variational inequalities (2) and thus dominates the shareholder value function. By construction, the function \bar{V} satisfies $\max_i D(i)\bar{V}(x) \leq 0$ and $\bar{V}' \geq 1$. We just have to check that $\bar{V}(x) \geq M\bar{V}(x)$ for $x \geq 0$ since by construction $\bar{V}(x) = \max(0, M\bar{V}(0))$ for $x < 0$. Because $\bar{V}' \geq 1$,

$$\begin{aligned} \bar{V}(x) &= \bar{V}(0) + \int_0^x \bar{V}'(y) dy \\ &\geq \bar{V}(0) + x \\ &= M\bar{V}(x). \end{aligned}$$

Proof of Proposition 5: The proof is similar to that of Proposition 2.

According to Jeanblanc and Shiryaev (1995), there is a unique pair $(\tilde{V}_0, \tilde{x}_0)$ solution to the following boundary problem:

$$\begin{cases} \frac{\sigma^2}{2}\tilde{V}_0'' + \mu\tilde{V}_0' - (r + \lambda)\tilde{V}_0 = 0 \text{ for } x \leq \tilde{x}_0 \\ \tilde{V}_0(0) = 0, \tilde{V}_0'(\tilde{x}_0) = 1 \text{ and } \tilde{V}_0''(\tilde{x}_0) = 0 \end{cases}$$

The function \tilde{V}_0 and the threshold \tilde{x}_0 are explicit and given by

$$\tilde{V}_0(x) = \begin{cases} \frac{e^{\bar{\theta}_2 x} - e^{\bar{\theta}_1 x}}{\bar{\theta}_2 e^{\bar{\theta}_2 \tilde{x}_0} - \bar{\theta}_1 e^{\bar{\theta}_1 \tilde{x}_0}} & \text{for } x \leq \tilde{x}_0 \\ x - \tilde{x}_0 + \frac{\mu}{r + \lambda} & \text{for } x > \tilde{x}_0, \end{cases} \quad (\text{A.9})$$

where

$$\tilde{x}_0 = \frac{1}{\bar{\theta}_2 - \bar{\theta}_1} \ln \frac{\bar{\theta}_1^2}{\bar{\theta}_2^2}. \quad (\text{A.10})$$

The only thing we have to check is that $D(1)\tilde{V}_0(x) \leq 0$ for every $x > 0$. First, for $x \geq \tilde{x}_0$, one has

$$D(1)\tilde{V}_0(x) = -(r + \lambda)(x - \tilde{x}_0) - \lambda L < 0.$$

On the other hand, on the interval $(0, \tilde{x}_0)$, one has

$$\begin{aligned} D(1)\tilde{V}_0(x) &= D(1)\tilde{V}_0(x) - D(0)\tilde{V}_0(x) \\ &= \tilde{V}_0(x) - L\tilde{V}_0'(x). \end{aligned}$$

Finally, note that the function $D(1)\tilde{V}_0(x)$ is increasing on $(0, \tilde{x}_0)$ since \tilde{V}_0 is concave and $D(1)\tilde{V}_0(\tilde{x}_0) = \frac{\mu}{r+\lambda} - L \leq 0$.

Proof of Proposition 6

Because $f \leq \frac{\mu}{r+\lambda} - \tilde{x}_0$ where \tilde{x}_0 is given by Equation(A.10), we can adapt the proof of Proposition 3 to show that there is a unique solution $(\tilde{V}_1, \tilde{x}_1)$ to the free boundary problem:

$$\begin{cases} \frac{\sigma^2}{2}\tilde{V}_1'' + \mu\tilde{V}_1' - (r + \lambda)\tilde{V}_1 = 0 & \text{for } x \leq \tilde{x}_1 \\ \tilde{V}_1(0) = \tilde{V}_1(\tilde{x}_1) - \tilde{x}_1 - f, \tilde{V}_1'(\tilde{x}_1) = 1 & \text{and } \tilde{V}_1''(\tilde{x}_1) = 0 \\ \tilde{V}_1(x) = \max(0, x + \tilde{V}_1(0)) & \text{for } x < 0 \end{cases} \quad (\text{A.11})$$

Proof of Proposition 7:

The idea again consists in building a smooth solution (V, x_0, x_1) to the following free boundary problem

$$\begin{cases} \frac{1}{2}\sigma^2 V''(x) + (\mu + r_0 x)V'(x) - (r + \lambda)V(x) = 0 & \text{for } 0 < x < x_0 \\ \frac{1}{2}\sigma^2 V''(x) + (\mu - \lambda L + r_0 x)V'(x) - rV(x) = 0 & \text{for } x_0 < x < x_1 \\ V(0) = 0 \quad V'(x_1) = 1, \quad V''(x_1) = 0. \end{cases} \quad (\text{A.12})$$

and applying the verification theorem (Theorem 2 in the appendix).

By standard results, we know that there is a unique solution H of the second order equation

$$\frac{1}{2}\sigma^2 H''(x) + (\mu + r_0 x)H'(x) - (r + \lambda)H(x) = 0$$

that satisfies the initial conditions $H(0) = 0$ $H'(0) = 1$. Our candidate solution V satisfies $V(x) = V'(0) \cdot H(x)$ for $0 < x < x_0$, where x_0 satisfies $V(x_0) = LV'(x_0)$ (since V is assumed to be smooth) or equivalently $H(x_0) = LH'(x_0)$.

Let $z = \inf\{x > 0, H''(x) \geq 0\}$. It is easy to see that z is finite and that H is concave on $(0, z)$. Now, two cases have to be considered.

1. $L > \frac{\mu + r_0 z}{r + \lambda}$.

In that case, it is easy to see that the function $\theta(x) = H(x) - LH'(x)$ is increasing with $\theta(z) = H(z) - LH'(z) < 0$. Therefore θ is negative also on $(0, z)$. the shareholder value function satisfies

$$V(x) = \begin{cases} \frac{H(x)}{H'(z)} & \text{for } 0 < x < z \\ x - z + \frac{\mu + r_0 z}{r + \lambda} & \text{for } x > z. \end{cases}$$

The optimal policy is to not buy any insurance and to accumulate cash up to threshold z .

2. $L < \frac{\mu+r_0z}{r+\lambda}$.

In that case, there is some $x_0 < z$ such that $\theta(x_0) = 0$. Now, for $x > x_0$, the shareholder value function V satisfies

$$V(x) = V(0)H'(x_0)(LK_0(x - x_0) + K_1(x - x_0)),$$

where K_0 and K_1 are the fundamental solutions of the second order ODE:

$$\frac{1}{2}\sigma^2 K''(x) + (\mu + r_0x - \lambda L)K'(x) - rV(x) = 0,$$

i.e. the solutions that satisfy the initial conditions: $K_0(0) = 1, K'_0(0) = 0, K_1(0) = 0$ and $K'_1(0) = 1$. It is easy to see that the function $\varphi(y) = LK_0(y) + K_1(y)$ admits an inflection point z_1 , in $0, z$. Thus $\varphi'' < 0$ on $(0, z_1)$ while $\varphi'' > 0$ on (z_1, z) . Setting $x_1 = x_0 + z_1$, we get the optimal solution as

$$V(x) = \begin{cases} \frac{H(x)}{H'(x_0)\varphi(z_1)} & \text{for } 0 < x < x_0 \\ \frac{\varphi(x-x_0)}{\varphi(z_1)} & \text{for } x_0 < x < x_1 \\ x - x_1 + \frac{\mu+r_0x_1}{r+\lambda} & \text{for } x > x_1. \end{cases}$$

In that case, the optimal policy is to insure whenever $x \geq x_0$ and to accumulate cash up to x_1 .

Proof of Proposition 8: Let H_0 and H_1 be the solutions of the ordinary differential equation $A(1, 0)H(x) = 0$ with initial conditions

$$H_0(0) = 1, H'_0(0) = 0, H_1(0) = 0, H'_1(0) = 1.$$

The condition $\hat{V}(0) = 0$ implies that $\hat{V}(x) = \hat{V}'(0)H_1(x)$ for $x \leq x_0$ where x_0 satisfies

$$H''_1(x_0) + \pi_h H'(x_0) = 0.$$

Under the assumption $\pi_h \leq \frac{2\mu}{\sigma^2 + \sigma_R^2}$, the function $\theta(x) = H''_1(x) + \pi_h H'(x)$ vanishes at some point \hat{x}_0 lower than the inflection point of H_1 .

Let K_0 and K_1 be the solutions of the ordinary differential equation $A(0, 0)f(\hat{x}_0 + y) = 0$ with initial conditions

$$K_0(0) = 1, K'_0(0) = 0, K_1(0) = 0, K'_1(0) = 1.$$

For $x \geq x_0$, we have

$$\hat{V}(x) = \hat{V}'(0)H'_1(\hat{x}_0)(\gamma K_0(x - \hat{x}_0) + K_1(x - \hat{x}_0)),$$

with $\gamma = \frac{\mu+r_0\hat{x}_0 - \frac{\pi_h}{2}(\sigma^2 + \sigma_R^2)}{r+\lambda}$. By the same argument as in Section 4.2, it is straightforward to prove that the function

$$\varphi(y) = \gamma K_0(y) + K_1(y)$$

admits an inflection point z . Set $\hat{x}_1 = \hat{x}_0 + z$ to conclude.

Proof of Proposition 9: As usual, we have to check that the solution \hat{V} of the Proposition 8 satisfies the verification theorem. By construction, we have $\hat{V}'(x) \geq 1$ for every $x > 0$ and

$A(h, 0)\hat{V}(x) \leq 0$ for $x \leq \hat{x}_1$.

Now, for $x \geq \hat{x}_1$, we have

$$\begin{aligned}
A(h, 0)\hat{V}(x) &= \lambda\hat{V}(x - L) - h\frac{\sigma_R^2}{2}\pi_h - \lambda(x - x_1) - (r - r_0)(x - \hat{x}_1) \\
&\leq \lambda(\hat{V}(\hat{x}_1) + x - L - \hat{x}_1) - h\frac{\sigma_R^2}{2}\pi_h - \lambda(x - \hat{x}_1) - (r - r_0)(x - \hat{x}_1) \\
&= \lambda\left(\frac{\mu + r_0\hat{x}_1}{r + \lambda} - L\right) - h\frac{\sigma_R^2}{2}\pi_h - (r - r_0)(x - \hat{x}_1) \\
&\leq 0,
\end{aligned}$$

since $L \geq \frac{\mu + r_0\hat{x}_1}{r + \lambda}$.

It remains to show that $A(h, 1)\hat{V}(x) \leq 0$ for every $x > 0$ and every h . For $x \geq \hat{x}_1$, we directly have

$$A(h, 1)\hat{V}(x) = \lambda\left(\frac{\mu + r_0\hat{x}_1}{r + \lambda} - L\right) - h\frac{\sigma_R^2}{2}\pi_h - (r - r_0)(x - \hat{x}_1) \leq 0,$$

while for $x < \hat{x}_1$,

$$\begin{aligned}
A(h, 1)\hat{V}(x) &= (A(h, 1)\hat{V}(x) - A(h, 0)\hat{V}(x)) + A(h, 0)\hat{V}(x) \\
&\leq A(h, 1)\hat{V}(x) - A(h, 0)\hat{V}(x) \\
&\leq \lambda(\hat{V}(x) - L\hat{V}'(x)).
\end{aligned}$$

But, the function $\hat{V}(x) - L\hat{V}'(x)$ is clearly increasing on $(0, \hat{x}_1)$ and negative at \hat{x}_1 . This ends the proof.

Proof of Proposition 10: The cost of financial frictions is given by:

$$x^*(\mu, \sigma^2) = \frac{1}{\theta_2 - \theta_1} \ln \frac{\theta_1^2}{\theta_2^2} = \frac{\sigma^2}{2\sqrt{\mu^2 + 2r\sigma^2}} \ln \left[\frac{\sqrt{\mu^2 + 2r\sigma^2} + \mu}{\sqrt{\mu^2 + 2r\sigma^2} - \mu} \right].$$

x^* is a continuous, positive function of μ satisfying

$$\lim_{\{\mu \rightarrow 0\}} x^*(\mu, \sigma^2) = \lim_{\{\mu \rightarrow \infty\}} x^*(\mu, \sigma^2) = 0.$$

A straightforward but tedious computation gives

$$\frac{\partial x^*}{\partial \mu}(\mu, \sigma^2) = \sigma^2(\mu + 2r\sigma^2)^{-\frac{3}{2}} \left[-\mu \operatorname{Argtanh} \left(\frac{\mu}{\sqrt{\mu^2 + 2r\sigma^2}} \right) + \sqrt{\mu^2 + 2r\sigma^2} \right],$$

where $\operatorname{Argtanh}$ is the inverse function of the hyperbolic tangent function.

Thus $\frac{\partial x^*}{\partial \mu}$ has the sign of

$$g(\mu) = -\mu \operatorname{Argtanh} \left(\frac{\mu}{\sqrt{\mu^2 + 2r\sigma^2}} \right) + \sqrt{\mu^2 + 2r\sigma^2}.$$

But, $g'(\mu) = -\operatorname{Argtanh} \left(\frac{\mu}{\sqrt{\mu^2 + 2r\sigma^2}} \right) < 0$, while $g(0) = \sqrt{2r\sigma^2}$ and $\lim_{\{\mu \rightarrow \infty\}} g(\mu) = -\infty$.

Therefore, g changes sign exactly once and x^* is uniquely defined.

Moreover, setting $t = \frac{\mu}{\sigma^2}$ and $a = \frac{2r}{\mu}$, we get

$$x^*(\mu, \sigma^2) \equiv f(t) = \frac{1}{\sqrt{t^2 + a}} \operatorname{Argtanh} \left(\sqrt{\frac{t}{t+a}} \right).$$

We have $f'(0) = -1$ and $f''(t) = -2 \operatorname{Argtanh} \left(\sqrt{\frac{t}{t+a}} \right)$. Therefore,

$$\frac{\partial x^*}{\partial \sigma^2} = \frac{\partial t}{\partial \sigma^2} f'(t) \geq 0.$$

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